NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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INFINITESIMAL CONICAL SUPERSONIC FLOW

By Adolf Busemann

Translation

Infinitesimalo kogelige Uberschallstromung

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Conical flow fields.- Real flows always occur in three-dimensional space. In calculating a flow, however, one will greatly appreciate it if there are only two essential coordinates to deal with. Flows of this kind, limited to two coordinates, form the plane flow and the flow of axial symmetry. The space which is filled out by the streamlines is represented in planes parallel to these lines; they contain certain streamlines to their whole extent. In conical flow fields, however, the streamlines are cut through slantingly so that each streamline is contained in the plane but appears there as a point only. These relations are made clear in figure 1. If the friction is neglected the shape of the body leads one to expect a pattern that can be increased or decreased geometrically. The fixed point P and the direction of the three spatial axes, x, y, and z remain the same. All essential characteristics of the flow and the shape of the body can be inferred from the plane $z = 1$. A plane $z = 2$ would, if distances were doubled, show identical values for gas conditions and velocities. The isobar planes in the space x, y, z are of conical shape and have the cone vertex P; therefore these flows shall be called abbreviatedly conical flow fields.

Infinitesimal differences in pressure.- In figure 1 there shall be one more limitation for the general conical flow field, namely that the body disturbs the parallel flow only to a slight degree. So the conical isobars reflect over- and under-pressures differing infinitesimally from the pressure of the parallel flow. This twofold limitation, to conical and infinitesimal, is not actually very stringent insofar as in the class of potential flows there are present only the conical fields of axial symmetry and the infinitesimal conical fields. All other conical flows are affected by rotation. The infinitesimal supersonic flows, however, also excel in another way: the superposition of fields with different fixed points P is permitted in spite of the fact that the differential equations ordinarily are not linear; thus the applicability is broadened most gratifyingly.

*"Infinitesimale kegelige Überschallströmung." Deutschen Akademie der Luftfahrtforshung, 1942-43, p. 455.

Differential equation.- There are two ways to limit the differential equation for the potential in conical fields to small additional velocities u, v, w, and to limit the differential equation for nearly parallel spatial flows to conical fields; the first is the historical one, the second, however, the simpler one. Therefore here the second one is chosen. As is well known, the linearized differential equation in the space x, y, z, for the additional potential φ over a basic velocity W in the direction of the axis z reads if the gas has the sonic velocity a:

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz}\left(1 - \frac{W^2}{a^2}\right) = 0 \tag{1}$$

The coordinates ξ and η of the conical current correspond to the spatial coordinates x and y in the plane $z = 1$:

$$\xi = \frac{x}{z} \quad \text{and} \quad \eta = \frac{y}{z} \tag{2}$$

The additional potential φ increases on each ray through the fixed point P in proportion to the distance. Therefore, the potential divided by z is invariant on a single ray, and provides the potential of the conical flow:

$$\chi(\xi,\eta) = \frac{1}{z}\,\varphi(x,y,z) \tag{3}$$

The additional velocities u, v, w are the derivatives of the former potential.

$$\left.\begin{aligned} u &\equiv \varphi_x = \chi_\xi \qquad v \equiv \varphi_y = \chi_\eta \\ w &\equiv \varphi_z = \chi - \xi \cdot \chi_\xi - \eta \cdot \chi_\eta \end{aligned}\right\} \tag{4}$$

The differential equation for the new potential χ is determined from the old differential equation, and one obtains:

$$\left.\begin{aligned}\chi_{\xi\xi}\left(1-\frac{\xi^2}{A^2}\right)+\chi_{\eta\eta}\left(1-\frac{\eta^2}{A^2}\right)-\chi_{\xi\eta}\frac{2\xi\eta}{A^2}=0\\ \text{with}\qquad \frac{1}{A}=\sqrt{\frac{W^2}{a^2}-1}=\frac{1}{\operatorname{tg}\alpha}\end{aligned}\right\}\qquad(5)$$

It certainly is gratifying to recognize in the type of this differential equation an old acquaintance from the plane gas flow: for the stream function of the plane flow transformed according to Legendre and superimposed over the components of the current density produces exactly the same differential equation. In ordinary gases, however, the denominator A^2 is a local function; but there is a special gas with rectilinear adiabatics in the pressure-volume-diagram in which, as required, this denominator also remains constant. This gas is a special favorite where it is a mere question of numerical calculations.

Regions of influence.- The spatial differential equation of the gas flow at supersonic velocity is of hyperbolic character, as shown in equation (1). That means: each point of the flow dominates a conical range opening downstream; each locus, on the other hand, is dominated solely by those points which are situated in the cone prolonged backward and opening upstream. Herewith the relationships are divided definitely among the three possibilities: superior, subordinate, and independent. Mach's cones in the supersonic flow considered as regions of disturbance of a small trial body make this fully comprehensible in a physical sense. It must seem odd at first that the dependencies of the general spatial flow are widened as soon as one proceeds to a more limited spatial flow. But the above-mentioned differential equation shows that inside of the circle with the radius A there prevails the elliptic character.

This behavior is easily explained by the fact that all points of a ray starting from P are comprehended as a whole. The relation of dependencies of two rays results from the dependencies of the single points. Only the characteristic "independent" appears uniformly in certain cases for all pairs of points (P itself is excluded). The combination superior and independent

becomes superior; subordinate and independent become subordinate. But if there are pairs of points of all kinds on the rays, then the rays are subject to the new characteristic "reciprocally dependent." Rays of this kind fill out the interior of Mach's cone starting from point P.

Characteristics.- Mach's cone starting from P intersects the plane $z = 1$ on the circle having the radius A. In the field outside of this cone, i. e., outside of the circle in the intersecting plane, one gets rectilinear characteristics of the differential equation (5) which are tangents of the circle. In figure 2 this is demonstrated by two wires, a and b. The wire b is bent slightly upstream in order not to exclude cases of this kind. The range of disturbance results from the sum of all of Mach's cones starting from all points of the wire. It is immediately obvious that only the circle with the radius A and its tangents can form the boundaries of the area of disturbance. Outside of Mach's cone starting from point P these characteristics settle all questions; they can be traced back to the plane case with a transverse component of the velocity. The essential and different part of the conical fields, therefore, is concerned with the convex surface of Mach's cone starting from point P, and with its interior.

Tschapligin's illustration.- In the plane of intersection $z = 1$ we find inside of the circle with the radius A the elliptic character of the differential equation (5). Near the center the differential equation of the potential theory is valid; in plane cases, this equation can be satisfied by analytic functions of the complex variable. In this circle, therefore, there only exists a mutual dependence but not yet a full equivalence of all loci. This is not surprising, because the analytic continuation of the plane reaches to the outer range of the circle. Tschapligin, however, has devised a geometrical construction which so distorts the field inside the circle that equivalence regarding the differential equation will result. As figure 3 shows this distortion is attained by transferring the plane $z = 1$, with the complex variable $\zeta = \xi + i\eta$, through parallel projection to a sphere with the radius A, and by then projecting it from a pole of the sphere on to a plane in the distance. One will easily recognize that only the interior of the circle with the radius A will be

depicted; first it will be delineated from the lower half-sphere on the interior of the unit circle of the new plane with the new complex variable ϵ; a second time it will go from the upper half-sphere on to the outer field of the unit circle. In these coordinates one can use analytic functions for the solutions.

SOLUTION OF THE DIFFERENTIAL EQUATION

For each of the velocity components u, v, and w one can equate the real part of an analytic function $f(\epsilon)$. It will serve the purpose best to set up the equation for the component w, because then the more closely related components u and v can be calculated jointly:

$$w = A \cdot \mathrm{Re}\,(f(\epsilon)) \quad \text{or} \quad w + is = A \cdot f\,(\epsilon) \tag{6}$$

The completion represented here by s is for the time being completely meaningless. According to Tschapligin there then results the complex velocity:

$$\omega = u + iv = -\frac{1}{2}\int\left(\frac{d\bar{f}}{\bar{\epsilon}} + \epsilon df\right) \tag{7}$$

The pressure in the current, with the aid of the density ρ, results from the velocity components as follows:

$$\left.\begin{aligned} p &= -\rho\left[W \cdot w + \frac{u^2 + v^2 + w^2}{2}\right] \\ &\approx -\frac{1}{2}\rho\left[AW\,(f + \bar{f}) + \omega\bar{\omega}\right] \end{aligned}\right\} \tag{8}$$

The right function $f(\epsilon)$ is to be selected with the aid of the boundary conditions.

<u>Boundary conditions</u>.- The outside of Mach's cone is superior to Mach's cone itself. Therefore, first, those velocities u, v, and w on the circle of the ζ-plane (and therefore on the uniform circle of the ϵ-plane) that result from the outer field must be ascertained.

If the body does not protrude anywhere out of Mach's cone, the values $u = v = w = 0$ on the uniform circle are given. If on the contrary no part of the body is inside of Mach's cone, the values of w are to be represented by an analytic function $f(\epsilon)$ free of singularities with the given boundary values on the circle. If $f(\epsilon)$ does not produce a stationary value $df = 0$, then u and v according to equation (7) will have a logarithmic singularity at zero. The many-leaved function can be selected in a unique way by using radial intersections with the boundary values of u and v on the uniform circle. The radial intersections produce rotational layers, as is physically to be expected from a lifting surface.

Impermeable boundaries of the body can be transferred into the ϵ-plane at the same time. They must be streamlines in the field of the relative velocity:

$$\Omega_{rel} = \omega + \epsilon^2\bar{\omega} - 2AW\epsilon \qquad (9)$$

This condition is not always easy to comply with. However, if the body possesses rectilinear surface elements passing near zero, the otherwise meaningless imaginary part s of the function $f(\epsilon)$ will remain constant on these elements. If the straight part goes over zero, a stationary value for f is to be stipulated at zero. Conditions of this kind are especially agreeable. From the pressure equation (8) conditions applicable to cases of given pressures or of given lifts are to be understood.

The disappearance of the real or of the imaginary part of f on certain lines because of symmetry can be attained in the well known way by reflexion, as the examples will show.

Examples

1. The circular cone in the straight flow

For the only axial-symmetrical case, i. e., the circular cone with an infinitesimal apex angle, the right solution is, of course, given by the statement

$$w + is = C \cdot \ln\epsilon$$

The pressure on the convex surface of the cone results in the known way (fig. 4) and conforms with v. Karman's values and mine.

2. The circular cone in oblique flow

One succeeds, with the aid of the relative velocity according to equation (9), in solving the circular cone in oblique flow. Herein the apex angle and the angle of attack may, though infinitesimal, yet bear a relation to each other. The solution is shown in figure 5. If one makes the angle of obliquity γ zero, one gets again the circular cone in straight flow. If one makes the apex angle 2β disappear, one gets the pressure distribution of a circular cone in an incompressible current. The comparison with Ferrari is rendered somewhat difficult by the fact that Ferrari measures the velocity field perpendicular to the cone axis while it is here perpendicular to the wind direction. If the system of coordinates is rotated adequately, the conformity is complete.

3. Tip of a rectangular plate

If a plane rectangular plate of infinitesimal thickness is placed in a flow perpendicular to the front edge with an infinitesimal angle of attack, and if the velocity field is needed only up to the rear edge of the plate, one can place the fixed point P at the right corner point of the front edge. On the supposition of an infinitesimal angle of attack γ (with the x-axis forming the axis of rotation) the pressure distribution will be represented on the quarter plane between the positive axis (z) and the negative axis (x). For the plane $z = 1$ the section of the body, except for infinitesimal distances, is then rendered by the negative real axis. Let the reduced pressure above the plate and the increased pressure below the plate be adjusted to a unit value outside of Mach's cone. These values hold on the boundary circle. On the left half of the unit circle of the ϵ-plane corresponding values for w are then to be assigned. On the right semicircle the outer field is undisturbed; here $w = 0$. For reasons of symmetry the value $w = 0$ must also result on the positive real axis. Along the negative real axis, on the contrary, $s = \mathrm{Im}(f(\epsilon))$ must be fixed because of the fixed radial boundary. Since s is given only up to one constant, one can demand here $s = 0$. All conditions can be

attained by reflexion if one undertakes a preliminary conformal mapping on the plane $\nu = \sqrt{\epsilon}$.

The solution is represented in figure 7. Figure 8 shows the pressure distribution on both edges of a rectangular plate.

4. Supporting triangle

Every two radii starting from P form a triangular plane as far as the plane $z = 1$, when all points are connected. Because of the required infinitesimal disturbance of the parallel current, however, the angle of attack must be infinitesimal, so that the plane of the two rays will nearly pass through the z-axis. Such triangles are possible completely inside of Mach's cone, completely outside, and uni- and bi-laterally protruding. Here we shall only consider the simplest case of the supporting triangle outside of Mach's cone, although all other cases can be easily integrated.

Figure 9 shows this supporting triangle. The velocity component w which predominantly influences the pressure is different from zero only on the short arcs between ζ_1 and ζ_3 as also ζ_4 and ζ_2. The value zero results from the undisturbed state on the right, and also on the left because of the pressure adjustment behind the triangle, when consideration is given to the symmetry with a positive and with a negative angle of attack. Figure 10 shows the relations in the ζ-plane. If one intends to let the rear edge of the triangle travel while the front edge lies fixed, one will at first transfer only the points ζ_1 and ζ_2 into the ϵ-plane. With suitable regulation there must result an increase of w from 0 to $+\pi$ at ζ_1, and from $-\pi$ to 0 at ζ_2 (if one moves on the circle in the direction of increasing angles). One can treat this part of the solution independently if one assumes a further singularity at zero. Physically speaking, one then has a uniformly loaded triangle between the front edge at ζ_{K_1} and the z-axis. Inside of Mach's cone, however, t is triangle is not flat, but is twisted to uniform or load. As soon as one superimposes at the rear edge a negatively loaded triangle and its influence between ζ_{K2} and the z-axis, the part behind the rear edge will no longer be supporting, and the singularity in w at the point zero of the ϵ-plane disappears.

However, a vortex layer in the field u, v is left. The partial solution in the ϵ-plane is represented in figure 11.

5. Superposition of two conical flows

The infinitesimal conical flows can be superposed without having the fixed point in common as in figure 4. Therefore the relations in the plane plate can also be represented when the plate has more depth. Figure 12 shows the isobars of the edge of the plate and also their superposition after Mach's cones have overlapped. Different boundary conditions for the partial solutions need be considered only when the cones reach the other edge of the plate. The disappearance of pressure along a straight line in just the distance at which the cones arrive at the other edge of the plate is remarkable. The positively loaded part of the plate ends here. Figure 13 shows the lift distribution of the positively loaded part in perspective representation.

To find the velocity field behind a rectangular plate of finite depth one can annul the supporting pressure differences of a plate of infinite depth by conical fields having the apices on the rear edge of the plate. If one superposes a negatively loaded plate shifted infinitesimally in the direction of the z-axis one obtains a supporting line as a limiting case of the supporting strip. The cases calculated by Schlichting according to Prandtl's method are obtained in this way. Here, too, the conformity is perfect, except for an error of sign in the calculation of the integral equation.

SUMMARY

The calculation of infinitesimal conical supersonic flows has been applied first to the simplest examples that have also been calculated in another way. Except for the discovery of a miscalculation in an older report there was found the expected conformity. The new method of calculation is limited more definitely to the conical case; but, as a compensation, it is much more convenient because the solution is obtained by analytic functions. The fundamental recognition that there the hyperbolic character is replaced by the elliptic one will lead to

more thorough investigation of conical fields as special cases in supersonic flows. Of course, one will be tempted to call the elliptic character seemingly elliptic only; for if one notches an indentation into a cone, the real hyperbolic character of the field of the flow downstream from this indentation immediately becomes obvious again. However, if one traces the flow to a point very far behind the indentation there will appear inside of Mach's cone a reciprocal relation between every two rays which will gradually restore the conical course of the flow. Instead of trying to produce the final flow by an infinite succession of hyperbolic dependencies it will be more expedient to consider special elliptic singularities at the points of disturbance. In these relations I see the significance of the conical field; the infinitesimal case represents only a first approximation to it.

Translation by Mary L. Mahler
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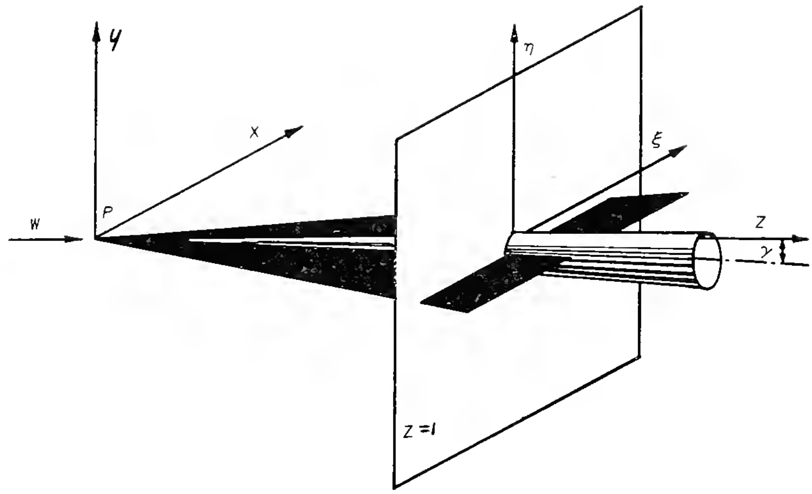


Figure 1. Coordinates in conical field

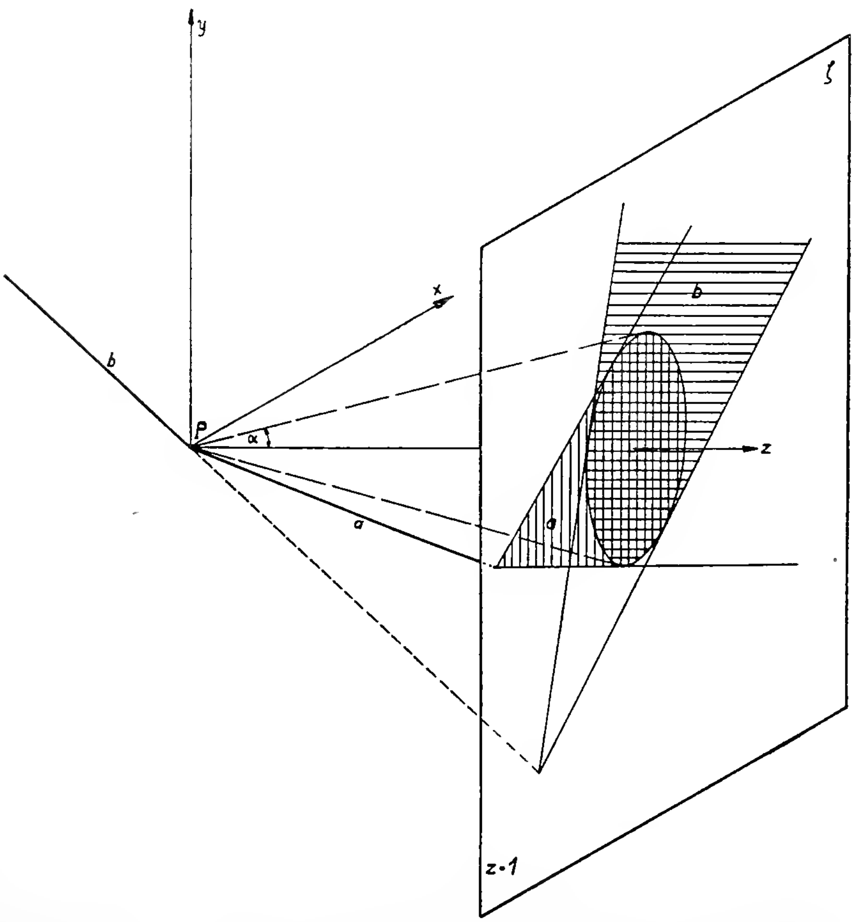


Figure 2. Disturbance field of elements a and b

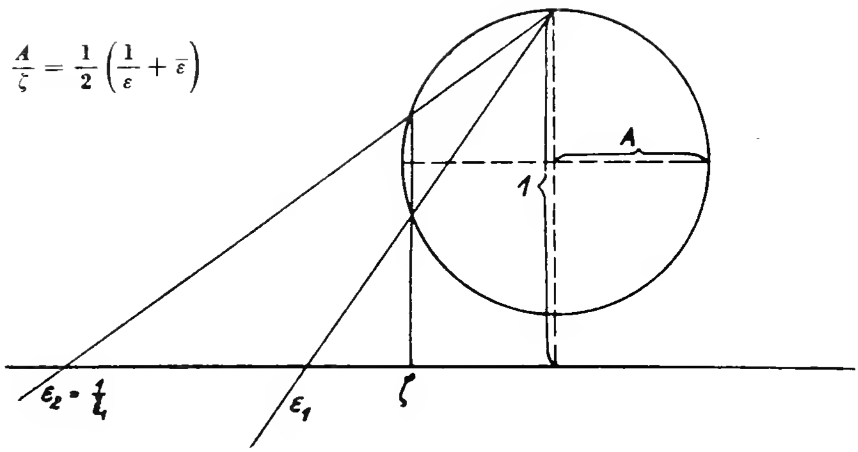


Figure 3. Chaplygin's transformation

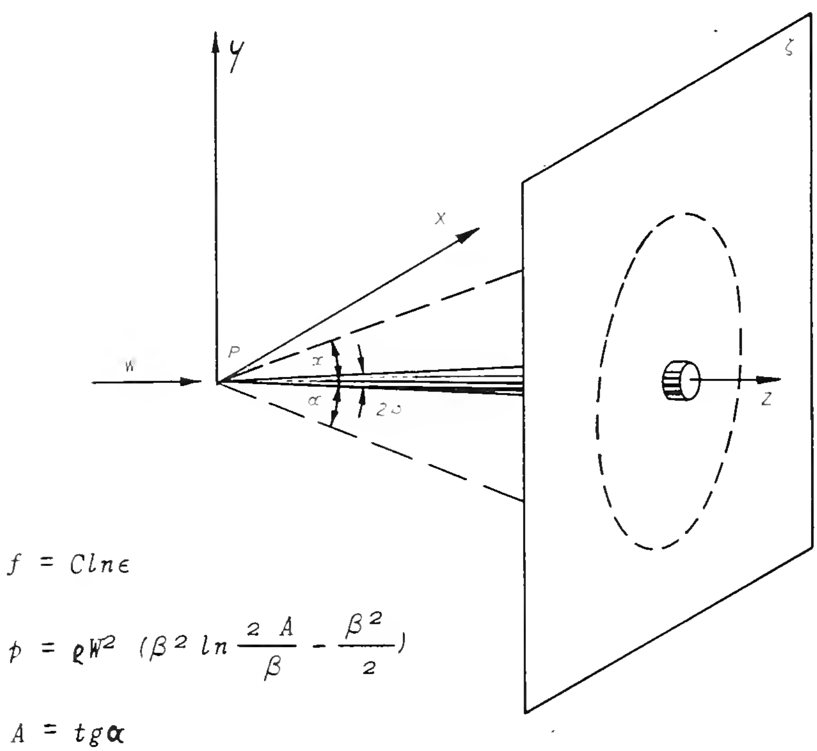
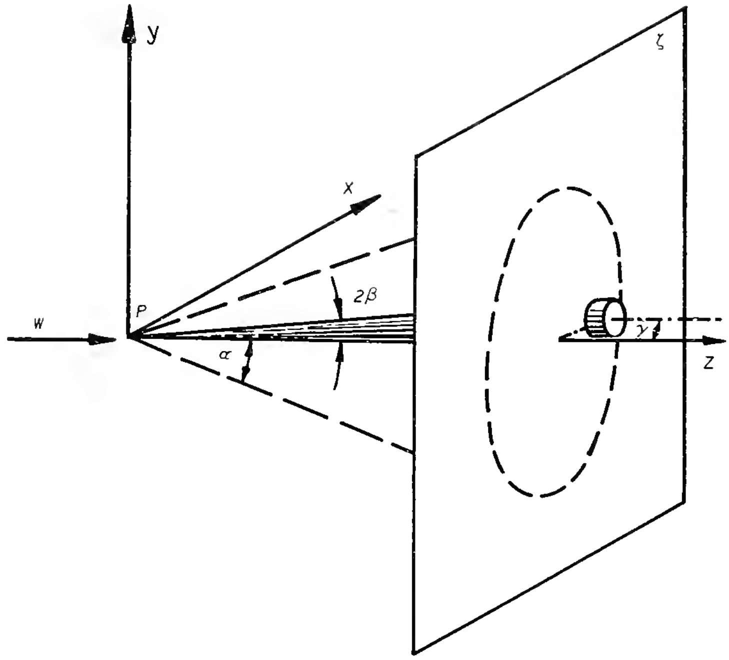


Figure 4. Circular cone in axial flow.

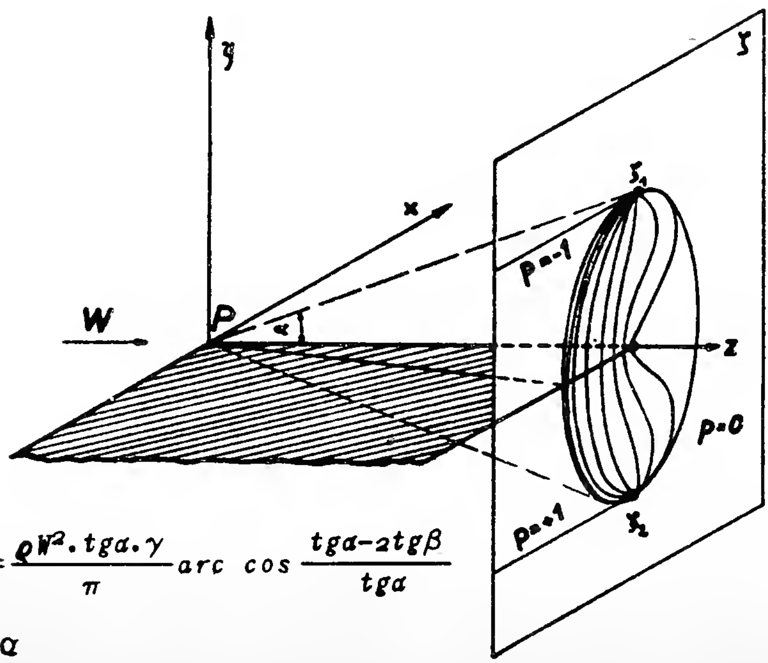


$$f = 4WAR^2 \left\{ ln \frac{\epsilon - \epsilon_0}{\epsilon - \frac{1}{\epsilon_0}} \quad \frac{1 - \frac{1}{\epsilon_0}}{1 - \epsilon_0} - \frac{3\epsilon_0}{\epsilon - \epsilon_0} + \frac{3\epsilon\epsilon_0}{1 - \epsilon\epsilon_0} - \frac{\epsilon_0^2}{(\epsilon - \epsilon_0)^2} + \frac{\epsilon_0^2 \ \epsilon^2}{(1 - \epsilon\epsilon_0)^2} \right\}$$

$$p = \varrho W^2 \left(\beta^2 \ ln \ \frac{2A}{\beta} - \frac{\beta^2}{2} - \frac{\gamma^2}{2} + 2\beta\gamma cos \ \delta + \gamma^2 \ cos \ 2\delta \right)$$

$$V = tg \, \alpha$$

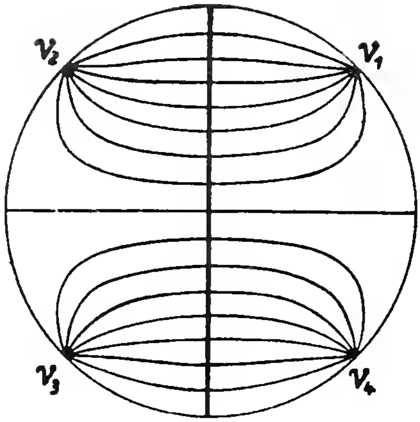
Figure 5. Circular cone in yawed flow



$$\frac{1}{2} (pd - ps) = \frac{\varrho W^2 . tg\alpha . \gamma}{\pi} arc \ cos \frac{tg\alpha - 2tg\beta}{tg\alpha}$$

$$A = tg \, \alpha$$

Figure 6. Edge of a rectangular plate



$$\nu = \sqrt{\epsilon}$$

$$f(\nu) = -i\left\{ \ln(\nu-\nu_1) - \ln(\nu-\nu_2) - \ln(\nu-\nu_3) + \ln(\nu-\nu_4) \right\}$$

Figure 7. Conformal representation at edge of a rectangular plate

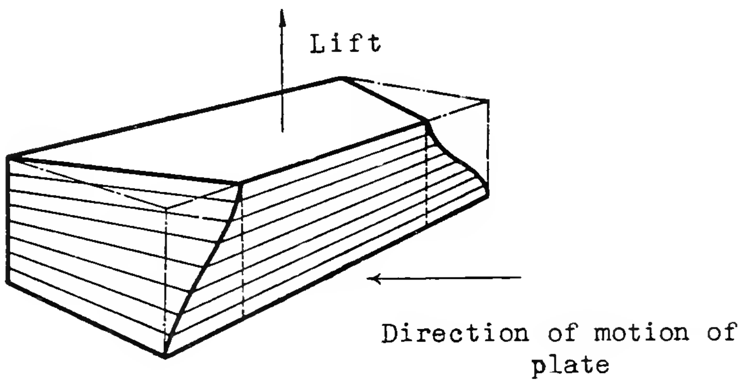


Figure 8. Pressure distribution on a flat plate

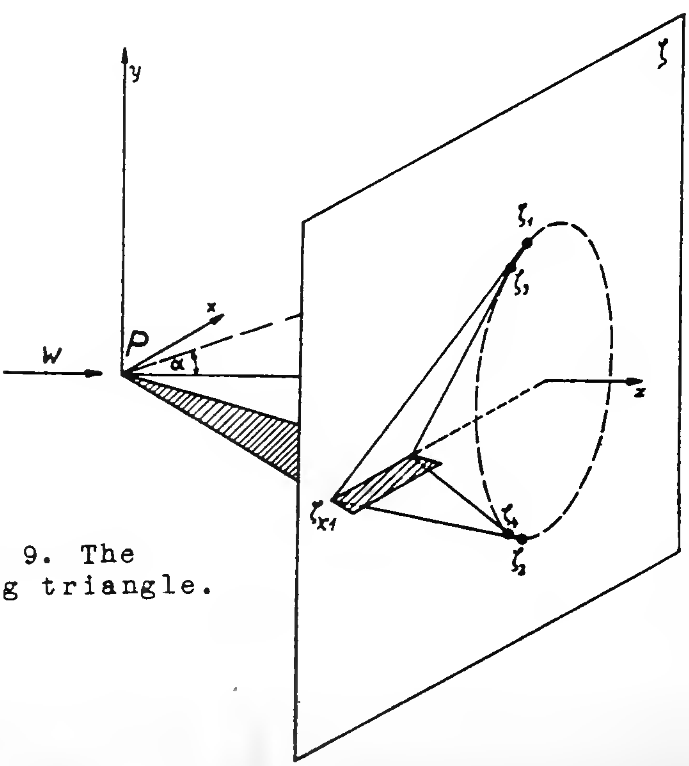
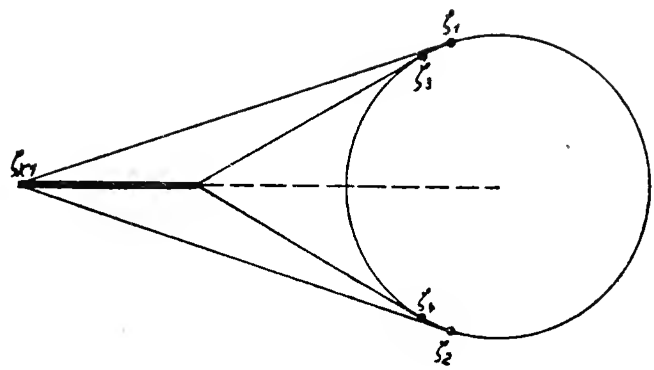


Figure 9. The lifting triangle.



$$\zeta = \frac{2A\epsilon}{1+\epsilon\bar{\epsilon}}$$

Figure 10. Cross section of the lifting triangle

$$f(\epsilon) = i\left\{ln(\epsilon-\epsilon_1)+ln(\epsilon-\epsilon_2)-ln\,\epsilon\right\}$$

$$pd-ps = const.$$

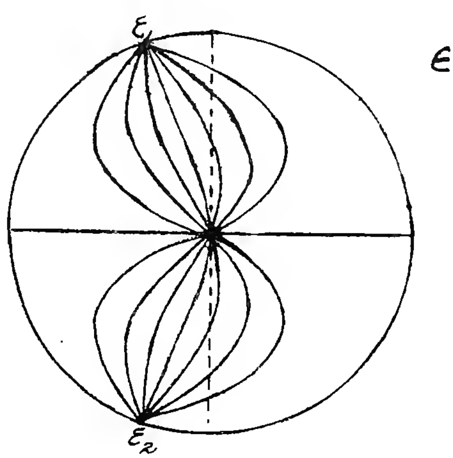


Figure 11. Representation of the lifting triangle in the ϵ plane

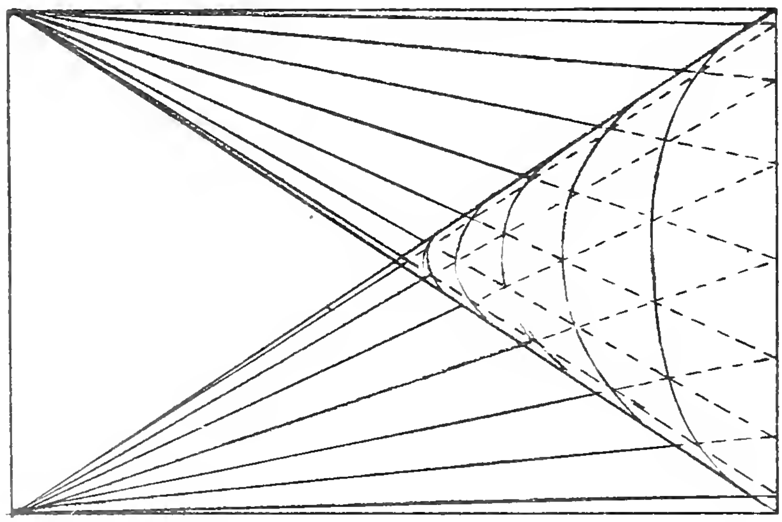


Figure 12. Superposition of edge influences for the rectangular plate at supersonic velocities

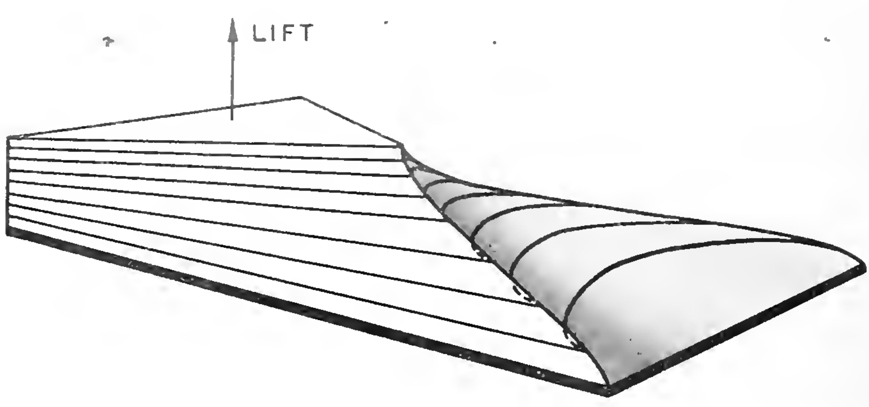


Figure 13. Pressure distribution on the rectangular plate at supersonic velocities